Quantum Mechanical State Preparation: A Simple Model

JAMES L. PARK† and WILLIAM BAND

Department of Physics,
Washington State University,
Pullman, Washington 99163

Abstract

Quantum dynamics is normally used to describe the causal evolution of the quantum state of a physical system between initial acts of *preparation* and final *measurement* operations. But logical consistency and completeness require that the same dynamical principles be applicable to the physical description of the preparation and measurement processes themselves. There is an extensive literature of contributions to the quantum theory of measurement, but very little theoretical scrutiny has been given to the process of state preparation.

   This paper begins with a philosophical analysis of the concept of preparation in quantum physics, then develops a classification of preparation schemes into *selective* and *coercive* types. An exact mathematical model illustrating coercive preparation is presented in detail.

1. Introduction

   The celebrated philosophical subtleties of quantum mechanics stem from its peculiarly unique status as a theory that is at once fundamental and statistical. By *fundamental*, we mean merely that contemporary evidence forcefully suggests that the propositions of quantum mechanics may be regarded as logically anterior to all other physical theories, and as adequate for the thorough scientific description of all known phenomena. Obviously history indicates that such a state of affairs is transitory; nevertheless present-day physics does in fact regard quantum mechanics as a kind of pervasive theoretical base which today occupies the fundamental epistemological niche that once belonged to classical analytical mechanics.

   Yet quantum mechanics is a probabilistic theory; all of its causal statements refer only to probabilities for obtaining possible numerical results if and when acts of measurement are performed. Moreover, the theory is irreducibly probabilistic in the sense that its mathematical structure, as the

uncertainty theorem reveals, cannot accommodate any description of a physical system in which all 'attributes' are assigned with unit probability. From a classical perspective, and from that of the 'hidden variables' school, the latter would seem to imply that quantum theory cannot be fundamental, for probability was once regarded merely as a measure of ignorance as to the exact behavior of an underlying microcosm which a truly fundamental theory would presumably describe without recourse to the probability concept.

Thus when we adopt the philosophical stance of the quantum theorist—that the construct *probability* may be a primitive in a *fundamental* physical theory—certain difficulties arise in applying the new viewpoint, especially to pragmatic empirical situations for which the classical 'common-sense' world view seems to provide the better mirror of direct experience.

One aspect of the dilemma is represented by the classical limit problem of quantum mechanics, which challenges the probabilistic quantum theory to account for the fact that macroscopic processes admit of deterministic description. Another aspect involves the theme of a complex set of investigations known in the literature as the quantum theory of measurement, the purpose of which is to describe quantally the empirical procedures by which numerical data are elicited from microsystems via interaction with macrosystems.

To these two classes of fundamental quantum theoretical problems, we would add a third—to describe quantally the empirical procedures, called preparations of state, wherein microsystems are prepared, via interaction with macroscopic apparatus, for subsequent measurement. The present paper is offered as a contribution to this third, largely ignored, class of problems. (Recently we have also explored the related problem (Band & Park, 1970, 1971) of extracting from data the quantum state representation belonging to a given preparation.)

This is not the place to dwell upon the nature of preparation and measurement; for more lengthy philosophical analyses, the reader is referred to other publications (Margenau, 1963; Park & Margenau, 1968). However, in order to establish clearly the meanings we have adopted for these often confused terms, we present a brief outline of what might be called the *preparation-measurement format* of experimental science.

(a) Because the scientific method eschews solipsism, only inter-subjective, hence *reproducible*, data are regularized by scientific theory. Hence the operational schemes by which systems are prepared for empirical investigation must be repeatable.

(b) Since quantum mechanics is irreducibly probabilistic, the epistemic link between its abstract constructs and empirical data is through statistics. For the operational definition of probability, the principle by which abstract probabilistic theorems are translated into testable assertions, is the following statistical rule: the probability that event M occurs in an experimental arrangement Π equals the relative

frequency with which M occurs in an ensemble of identical experimental arrangements Π.

(c) From (a) and (b) we conclude that quantum mechanics is not a verifiable scientific theory unless procedures exist which will invariably generate any given preparation, Π.

Furthermore, unless quantum mechanics can consistently describe such procedures, the theory denies its only means of verification and is thus self-contradictory.

(d) A single experimental *run* consists of a preparation, Π, followed by a measurement act which yields a numerical datum. Verification or falsification of quantum mechanical probability assertions is then accomplished by statistical analysis, based on the rule given in (b), of data gathered in an ensemble of runs each having begun with the reproducible preparation, Π.

In the mathematical language of quantum theory, each empirically possible Π is assumed to be representable by a density matrix ρ. Hence the problem of state preparation which the present work explores may be reduced to the following theoretical question: can quantum mechanics describe an environment E for a system S such that the end result of the interaction of S and E is to leave S in a *preassigned* quantum state ρ which is independent of the original state of S prior to the interaction? An affirmative answer will assure that quantum mechanics is not self-contradictory and that, despite its irreducible probabilities, it can cope within its own distinctive logical nexus with the empirical concept of reproducible preparation so vital to experimental science.

2. *Selective Versus Coercive Preparations of State*

The most commonly discussed state preparation scheme is a method erroneously called selective 'measurement'; its paradigm is that stage of the Stern–Gerlach experiment which precedes detection and measurement at the photographic plate. In order to understand the experiment in genuinely quantum mechanical terms, it is necessary to refine the popular jargon which speaks of 'splitting the beam and accepting only that portion containing the desired quantum state.' The Stern–Gerlach selection procedure may be rigorously described as follows: when an atom in any given initial state is subjected to the proper physical environment, the resultant causal evolution leaves the atom in a rather unusual state in which the joint probability distribution for internal energy and center-of-mass position correlates the two random variables; hence a position measurement will determine, by inference, an energy datum. Energy data thus obtained will be distributed according to the theoretical energy distribution inherent in the given initial state.

Now, it happens that this Stern–Gerlach correlation can be exploited to effect a state preparation for any desired energy eigenstate. By placing

absorbers in those spatial regions where detection of the atom would yield an energy other than the one whose eigenstate is desired, we create an apparatus from which only atoms certain to yield the desired energy can emerge. Thus the apparatus prepares the desired eigenstate of the energy. The overall device is often colorfully referred to as a 'filter'; but this terminology is misleading, especially when the initial state is pure and hence in principle indivisible. This philosophical matter is more thoroughly treated elsewhere (Park, 1968).

Briefly, what actually occurs is that the initial state, upon interaction with the absorbers, is converted to a special mixture resolvable into sub-ensembles, one of which is described by the desired energy eigenstate. The unwanted components of the mixture are then somehow diverted or destroyed; hence the subensemble of atoms which survive this gauntlet will be characterized by the desired energy eigenstate. We shall refer to preparation procedures of this kind, which involve conversion of the original state to a mixture of specified structure followed by selection from that mixture, as *selective preparations*.

Unfortunately, selective preparation is not really as generalizable as it might seem. Indeed its success depends strongly on the possibility of finding some means for realizing the critical correlations which make the selection process practicable. The correlations required are in fact mathematically isomorphic to those in von Neumann's old theory of measurement, a topic which has inspired interminable controversy over the past four decades. Consequently it seems appropriate to seek an alternative class of preparation schemes which would not depend upon the crucial selection process.

Margenau, who was probably first (Margenau, 1937) to recognize clearly the essential distinction between preparation and measurement, has illustrated the preparation concept by noting that the ground state of an atom may be prepared simply by 'waiting,' relying upon spontaneous emission to effect the desired preparation. The same observation is the germinal idea of a theory of preparation due to Lamb (1969). In the latter, any desired pure state is prepared by placing the system in a potential well designed, if possible, so that the new ground state is the modulus of the desired wave function. After the spontaneous preparation of the ground state has occurred, the desired phase is obtained by careful impulsive perturbation. If it is not possible to devise a potential whose ground state is the desired wave function, still it is usually possible to find a potential such that *one* of the bound states is the desired function. Thus in principle the desired state may be prepared by placing the system in the well, performing a *selective preparation* of the desired state (an energy eigenfunction of the system when in the well), then 'turning off' the potential. This method presupposes of course that the correlating interactions required to perform a selection can be found.

Rather than depend in any way on the selection process, we wish to explore the possibility of driving systems into preassigned quantum states via purely dynamical interactions. That is, we envisage a process wherein a

system, no matter what its initial state, is compelled to evolve into some desired final state. Such a process would constitute a preparation procedure for the desired state. Moreover, *every* sample of the system subjected to the process would emerge in the desired state, no selection having been involved. We shall call such a state preparation scheme a *coercive preparation*.

The spontaneous emission processes mentioned above are examples of coercive preparation; an atomic system is driven to its ground state by interaction with the vacuum state of the electromagnetic field. Lamb's theory of preparation is thus a combination of coercive and selective preparations. The remainder of the present paper is devoted to a detailed study of purely coercive preparations; in particular, we shall demonstrate the possibility of coercively preparing states other than the ground state.

A purely classical example may be worth keeping in mind; viz., a water-cooling system designed to bring a small container to a predetermined temperature.

3. *Quantum State Evolution for an Open System*

The simple model of a coercive preparation to be presented in Section 4 will require following the temporal evolution of the state of a quantum system S_0 as it interacts sequentially with the systems E_1, E_2, ..., E_K, a collection to be called the *environment* E of S_0. S_0 will be referred to simply as the *system*. The totality $S_0 + E$, system plus environment, will be termed the *global system*.

The Hilbert space for the global system is the tensor product of the Hilbert spaces of the constituents:

$$\mathscr{H} = \mathscr{H}_0 \otimes \mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_K = \prod_{k=0}^{K} \mathscr{H}_k \tag{3.1}$$

Let H_0, $H_{k>0}$ denote the energy operators for S_0, E_k, respectively, and V_{0k} the interaction potential energy between S_0 and E_k. If the eigenvalue equation for H_k, $k = 0, \ldots, K$, is expressed as

$$H_k |Ed\rangle_k = E |Ed\rangle_k \tag{3.2}$$

where d is an index to allow for degeneracy in the H_k spectrum, the global space $\mathscr{H}$ is spanned by the set

$$\{|Ed\rangle_0 |E^{(1)} d^{(1)}\rangle_1 \ldots |E^{(K)} d^{(K)}\rangle_K\} \tag{3.3}$$

The quantum state of the global system at time t is represented by the global statistical operator ('density matrix'), for which we adopt the symbol capital rho: $P(t)$. According to quantum theory, $P(t)$ evolves unitarily, its causal development being generated by the global Hamiltonian. The density matrix $\rho_0(t)$ of the system S_0, on the other hand, does not in general evolve unitarily; its temporal development is determined from that of $P(t)$ by applying von Neumann's theory of composite systems (von Neumann, 1955).

For simplicity in notation, but without loss of generality, we consider in the present section the case $K = 2$. The global Hamiltonian which generates time translation is then given by

$$
\begin{aligned}
H &= H_0 + H_1 + H_2 + V_{01} + V_{02} \\
&\equiv H_0 \otimes 1_1 \otimes 1_2 + 1_0 \otimes H_1 \otimes 1_2 + 1_0 \otimes 1_1 \otimes H_2 \\
&\quad + V_{01} \otimes 1_2 + V_{02} \otimes 1_1
\end{aligned}
\tag{3.4}
$$

where we have tacitly assumed that the environment subsystems E_1, E_2 are not mutually interacting. (1_k denotes the identity in $\mathscr{H}_k$.) Causal evolution of the global density matrix is therefore expressed as follows ($\hbar = 1$):

$$
P(t) = T(t) P(o) T^\dagger(t)
\tag{3.5}
$$

where $T(t)$ satisfies

$$
HT(t) = i \frac{dT(t)}{dt}, \quad T(o) = 1
\tag{3.6}
$$

To study the time development of the system of interest, we must calculate the reduced density matrix $\rho_0(t)$, which completely describes the statistics of measurement results obtained by measuring observables associated exclusively with S_0, i.e. observables of the form $A \otimes 1_1 \otimes 1_2$.

According to the theory of composite systems, $\rho_0(t)$ is computed by using the 'partial trace' operation defined by

$$
\begin{aligned}
\rho_0(t) &= Tr_{12} P(t) \\
&\equiv \sum_{ab} {}_1\langle a|_2\langle b| P(t) |b\rangle_2 |a\rangle_1
\end{aligned}
\tag{3.7}
$$

where $\{|\rangle_k\}$ may be any orthonormal set spanning $\mathscr{H}_k$. In working below with expressions like (3.7), it will prove convenient to use the set

$$
\{|E^{(1)} d^{(1)}\rangle_1 |E^{(2)} d^{(2)}\rangle_2\}
$$

defined in (3.3).

We assume that at $t = 0$, S_0, E_1, and E_2 are completely uncorrelated. This means that the joint probability distribution for any three observables of the form $A \otimes 1_1 \otimes 1_2$, $1_0 \otimes B \otimes 1_2$, $1_0 \otimes 1_1 \otimes C$ is a product of three functions of one variable each. The initial global density matrix will then have the product form

$$
P(o) = \rho_0(o) \otimes \rho_1(o) \otimes \rho_2(o)
\tag{3.8}
$$

We further assume that when $t \in [0, t_1]$, only S_0 and E_1 interact and that when $t \in [t_1, t_2]$, $t_1 < t_2$, only S_0 and E_2 interact.

In other words, the global Hamiltonian has the form

$$
\begin{aligned}
H(t) &= [1 - \theta(t - t_1)] H(1) + [\theta(t - t_1) - \theta(t - t_2)] H(2) \\
&\quad + \theta(t - t_2) H(3),
\end{aligned}
\tag{3.9}
$$

where θ is the Heaviside unit step function and

$$H(1) \equiv H_0 + H_1 + H_2 + V_{01}$$
$$H(2) \equiv H_0 + H_1 + H_2 + V_{02}$$
$$H(3) \equiv H_0 + H_1 + H_2 \qquad (3.10)$$

Thus intuitively we expect the evolution operator to be

$$
\begin{aligned}
T(t) = &[1 - \theta(t - t_1)] \exp\left[-itH(1)\right] + [\theta(t - t_1) - \theta(t - t_2)] \\
&\times \exp\left[-i(t - t_1) H(2)\right] \exp\left[-it_1 H(1)\right] + \theta(t - t_2) \\
&\times \exp\left[-i(t - t_2) H(3)\right] \exp\left[-i(t_2 - t_1) H(2)\right] \\
&\times \exp\left[-it_1 H(1)\right] \qquad (3.11)
\end{aligned}
$$

That (3.11) is the correct evolution operator for our discontinuous global Hamiltonian may be verified by noting that $T(t)$ is continuous and satisfies (3.6):

$$
\begin{aligned}
i\frac{dT}{dt} = &[1 - \theta(t - t_1)] H(1) \exp\left[-itH(1)\right] + [\theta(t - t_1) - \theta(t - t_2)] H(2) \\
&\times \exp\left[-i(t - t_1) H(2)\right] \exp\left[-it_1 H(1)\right] \\
&+ \theta(t - t_2) H(3) \exp\left[-i(t - t_2) H(3)\right] \exp\left[-i(t_2 - t_1) H(2)\right] \\
&\times \exp\left[-it_1 H(1)\right] \\
&- i\delta(t - t_1) \exp\left[-itH(1)\right] + i[\delta(t - t_1) - \delta(t - t_2)] \\
&\times \exp\left[-i(t - t_1) H(2)\right] \exp\left[-it_1 H(1)\right] \\
&+ i\delta(t - t_2) \exp\left[-i(t - t_2) H(3)\right] \exp\left[-i(t_2 - t_1) H(2)\right] \\
&\times \exp\left[-it_1 H(1)\right] \\
= &\{[1 - \theta(t - t_1)] H(1) + [\theta(t - t_1) - \theta(t - t_2)] H(2) + \theta(t - t_2) H(3)\} \\
&\times \{[1 - \theta(t - t_1)] \exp\left[-itH(1)\right] + [\theta(t - t_1) - \theta(t - t_2)] \\
&\times \exp\left[-i(t - t_1) H(2)\right] \exp\left[-it_1 H(1)\right] + \theta(t - t_2) \\
&\times \exp\left[-i(t - t_2) H(3)\right] \exp\left[-i(t_2 - t_1) H(2)\right] \exp\left[-it_1 H(1)\right]\} \\
&- i\delta(t - t_1)\{\exp\left[-it_1 H(1)\right] - \exp\left[-i(t_1 - t_1) H(2)\right] \\
&\times \exp\left[-it_1 H(1)\right]\} \\
&- i\delta(t - t_2) \{\exp\left[-i(t_2 - t_1) H(2)\right] \exp\left[-it_1 H(1)\right] \\
&- \exp\left[-i(t_2 - t_2) H(3)\right] \exp\left[-i(t_2 - t_1) H(2)\right] \exp\left[-it_1 H(1)\right]\} \\
= &HT - i\delta(t - t_1)[0] - i\delta(t - t_2)[0] = HT
\end{aligned}
$$

From (3.11), we find the evolution operator for $t = t_2$ to be

$$T(t_2) = \exp\left[-i(t_2 - t_1) H(2)\right] \exp\left[-it_1 H(1)\right] \qquad (3.12)$$

Since the commutators $[H_0 + H_1 + V_{01}, H_2] = [H_0 + H_2 + V_{02}, H_1] = 0$, (3.12) may be partially factored:

$$
\begin{aligned}
T(t_2) = &\exp\left[-i(t_2 - t_1) H_1\right] \exp\left[-i(t_2 - t_1)(H_0 + H_2 + V_{02})\right] \\
&\times \exp\left[-it_1(H_0 + H_1 + V_{01})\right] \exp\left[-it_1 H_2\right] \qquad (3.13)
\end{aligned}
$$

Thus

$$P(t_2) = \exp\left[-i(t_2 - t_1)H(2)\right]\left[\rho_{01}(t_1) \otimes \rho_2(t_1)\right]\exp\left[i(t_2 - t_1)H(2)\right] \tag{3.14}$$

where

$$\rho_{01}(t_1) = \exp\left[-it_1(H_0 + H_1 + V_{01})\right]\left[\rho_0(o) \otimes \rho_1(o)\right]\exp\left[it(H_0 + H_1 + V_{01})\right] \tag{3.15}$$

and

$$\rho_2(t_1) = \exp\left(-it_1\,H_2\right)\rho_2(o)\exp\left(it_1\,H_2\right) \tag{3.16}$$

Note that $\rho_{01}(t_1)$ is the quantum state that would be predicted by unitary evolution prior to t_1 of $S_0 + E_1$, and $\rho_2(t_1)$ is just the freely evolved state of E_2 during the same interval.

We next extract $\rho_0(t_2)$ from $P(t_2)$:

$$\rho_0(t_2) = \mathrm{Tr}_{12}\,P(t_2)$$

$$= \mathrm{Tr}_2\left\{\sum_{Ed}{}_1\langle Ed|\exp\left[-i(t_2 - t_1)H_1\right]\exp\left[-i(t_2 - t_1)(H_0 + H_2 + V_{02})\right]\right.$$

$$\times\left[\rho_{01}(t_1) \otimes \rho_2(t_1)\right]\exp\left[i(t_2 - t_1)(H_0 + H_2 + V_{02})\right]$$

$$\left.\times\exp\left[i(t_2 - t_1)H_1\right]|Ed\rangle_1\right\}$$

$$= \mathrm{Tr}_2\left\{\exp\left[-i(t_2 - t_1)(E - E)\right]\exp\left[-i(t_2 - t_1)(H_0 + H_2 + V_{02})\right]\right.$$

$$\times\left[\sum_{Ed}{}_1\langle Ed|\rho_{01}(t_1)|Ed\rangle_1 \otimes \rho_2(t_1)\right]$$

$$\left.\times\exp\left[i(t_2 - t_1)(H_0 + H_2 + V_{02})\right]\right\}$$

$$= \mathrm{Tr}_2\left\{\exp\left[-i(t_2 - t_1)(H_0 + H_2 + V_{02})\right]\left[\rho_0(t_1) \otimes \rho_2(t_1)\right]\right.$$

$$\left.\times\exp\left[-i(t_2 - t_1)(H_0 + H_2 + V_{02})\right]\right\} \tag{3.17}$$

where $\rho_0(t_1) = \mathrm{Tr}_1\rho_{01}(t_1)$, the reduced density matrix for S_0 at t_1 that would be predicted by unitary evolution of $S_0 + E_1$, ignoring E_2. Equation (17) further demonstrates that subsequent evolution of $\rho_0(t)$ is correctly predicted by unitary evolution of $S_0 + E_2$, ignoring E_1, with $\rho_0(t_1)$ taken as the initial state of S_0 for the $S_0 - E_2$ interaction.

From the structure of (3.17) we therefore infer a general procedure for determining $\rho_0(t_K)$ after S_0 has interacted sequentially with $E_1, \ldots, E_K$:

(a) Find the unitary evolution of $S_0 + E_1$ until $t = t_1$; then obtain $\rho_0(t_1)$ by reducing the $S_0 + E_1$ density matrix.

(b) Take $\rho_0(t_1)$ as the initial state of S_0 and find the unitary evolution of $S_0 + E_2$ until $t = t_2$; obtain $\rho_0(t_2)$ by reduction of the $S_0 + E_2$ density matrix.

(c) Repeat the procedure until $\rho_0(t_K)$ is determined.

The main advantage of this procedure will be apparent below; the need to consider S_0 and only one environment subsystem E_k at a time greatly diminishes the bookkeeping inherent in any study of composite systems. However, it must be recognized that the procedure is not valid unless the conditions stated earlier are met; viz., S_0 and the E_k are all initially uncorrelated, and S_0 interacts with each E_k individually.

Finally, we wish to emphasize that the $\rho_0(t_K)$ derived by the prescription above is a rigorous dynamical result. At no point was the dubious and grotesque notion of acausal wave packet collapse invoked.

4. *An Illustration of Coercive Preparation*

4.1. *Description of the Global System*

Consider a system S_0 characterized by a two-dimensional Hilbert space $\mathcal{H}_0$ spanned by the eigenvectors $|E\rangle_0$, $|F\rangle_0$ of H_0. The environment E with which the system interacts during the coercive preparation process is a sequence of identical environment subsystems $\{E_1, \ldots, E_k, \ldots, E_K\}$ each of which has the same energy spectrum $\{E, F\}$ as S_0, except that both E and F are taken to be N-fold degenerate. Hence the typical E_k is characterized by a $2N$-dimensional Hilbert space $\mathcal{H}_k$ spanned by the eigenvectors of H_k: $\{|Ed\rangle_k, |Fd\rangle_k | d = 1, \ldots, N\}$.

The energy spectra of S_0 and E_k have been chosen the same to permit conservative energy exchanges between the system and each environment subsystem. The incorporation of a degree of degeneracy into the H_k spectrum was motivated by the fact that in actual practice the preparing environment would be macroscopic in character, and macroscopic systems typically have quite degenerate energy spectra.

To treat the interaction between S_0 and E_k, we must use the $4N$-dimensional Hilbert space spanned by all the tensor products of H_0 and H_k eigenvectors. To minimize the tedium inherent in manipulating quantities associated with such a space, we shall employ the following codes for the tensor products involved:

$$\left\{\begin{array}{l} |E\rangle_0|Ed\rangle_k \equiv |d\rangle \equiv |1d_1\rangle \\ |F\rangle_0|Ed\rangle_k \equiv |N+d\rangle \equiv |2d_1\rangle \\ |E\rangle_0|Fd\rangle_k \equiv |2N+d\rangle \equiv |1d_2\rangle \\ |F\rangle_0|Fd\rangle_k \equiv |3N+d\rangle \equiv |2d_2\rangle \end{array}\right\} \qquad (4.1.1)$$

where $d = 1, \ldots, N$. Note that the subscripts on the d's in the right-most symbols are keyed to energy levels of E_k. The subscript k has been dropped since the Hilbert space for each identical E_k has the same structure.

2

In terms of this code, the (m, n) matrix element of $H_0 + H_k$ is given by

$$(H_0 + H_k)_{mn} \equiv \langle m | H_0 + H_k | n \rangle$$
$$= \delta_{mn}\{2E[1 - \theta(m - N)] + (E + F)[\theta(m - N) - \theta(m - 3N)]$$
$$+ 2F\theta(m - 3N)\}, \quad 1 \leqslant m, n \leqslant 4N, \qquad (4.1.2)$$

where

$$\theta(l - L) \equiv \begin{cases} 0, & l \leqslant L \\ 1, & l > L \end{cases}. \qquad (4.1.3)$$

In what follows we shall sometimes use this θ symbol to describe matrix elements, as in (4.1.2); frequently, however, we simply indicate in brackets following a term in a matrix element expression the ranges of row and column indices for which that term is present in the matrix. For example, (4.1.2) might be expressed as follows:

$$(H_0 + H_k)_{mn} = \delta_{mn}\{2E[m \leqslant N] + (E + F)[N < m \leqslant 3N]$$
$$+ 2F[3N < m]\} \qquad (4.1.4)$$

To describe the interaction between S_0 and E_k, the following potential energy matrix will be assumed:

$$(V)_{mn} = v(1 - \delta_{mn})[N < m, n \leqslant 3N], \quad v \text{ real} \qquad (4.1.5)$$

There are several reasons why the form (4.1.5) was chosen for the interaction potential:

(a) V satisfies the commutation relation

$$[H_0 + H_k + V, H_0 + H_k] = 0 \qquad (4.1.6)$$

which means that the interaction merely transfers energy between S_0 and E_k. In particular, if $S_0 + E_k$ initially has a definite internal energy $(E + F)$, V promotes redistribution of that energy among S_0 and E_k but cannot alter the total energy. However, V as given by (4.1.5) is not uniquely determined by (4.1.6).

(b) V connects the two members of each pair of distinct $(H_0 + H_k)$ eigenstates belonging to $(E + F)$ with the same matrix element v. Hence the transition amplitude for each possible energy exchange between S_0 and E_k is the same. (We wished to avoid any quantal analogue of Maxwell's demon.) However, the assignment of equal probability to each such exchange does not uniquely specify the form (4.1.5), since equal transition probabilities require only equal moduli of the (complex) transition amplitudes.

(c) Both the choice of a real v and the use of zeros for V matrix elements insensitive to the conditions in (a) and (b) were motivated by the desire to solve the dynamical problem exactly, without recourse to time-dependent perturbation theory since the latter often introduces irrelevant mathematical complexities. (In retrospect, we found by comparison of the exact solution with the perturbation theory result

that a perturbation treatment, if carried through second order in V, does correctly indicate the general trend of the exact time development of $\rho_0(t)$.)

As explained in Section 3, it will be assumed that the initial state of the global system has the uncorrelated form

$$P(o) = \rho_0(o) \otimes \rho_1(o) \otimes \ldots \otimes \rho_K(o) \qquad (4.1.7)$$

and that the system interacts sequentially with each environment subsystem. Since we are describing a coercive preparation of S_0, $\rho_0(o)$ is left arbitrary. However, the initial state of the preparing environment must be chosen such that subsequent evolution of the global system will produce the desired final state ρ_0 of S_0. We shall assume that each initial state $\rho_k(o)$ is the density matrix of a microcanonical ensemble with energy F; i.e., each (macroscopic) environment subsystem is in a state of thermodynamical equilibrium with energy F. The matrix elements of $\rho_k(o)$, $k > 0$, are therefore given by

$$\left\{ \begin{matrix} {}_k\langle Ed|\rho_k(o)|Ed'\rangle_k = {}_k\langle Ed|\rho_k(o)|Fd'\rangle_k = {}_k\langle Fd|\rho_k(o)|Ed'\rangle_k = 0 \\ {}_k\langle Fd|\rho_k(o)|Fd'\rangle_k = \dfrac{1}{N}\delta_{dd'} \end{matrix} \right\} (4.1.8)$$

Because the microcanonical ensemble is a stationary state,

$$\rho_k(t) = \rho_k(o), \quad t < t_{k-1} \qquad (4.1.9)$$

since E_k evolves freely except during its interval $[t_{k-1}, t_k]$ of interaction with S_0.

We shall prove below that the system S_0, when subjected to the sequence of interactions with environment subsystems just described, will invariably be driven to the final state given by

$$\left\{ \begin{matrix} {}_0\langle E|\rho_0|E\rangle_0 = {}_0\langle E|\rho_0|F\rangle_0 = {}_0\langle F|\rho_0|E\rangle_0 = 0 \\ {}_0\langle F|\rho_0|F\rangle_0 = 1 \end{matrix} \right\} \qquad (4.1.10)$$

(Note that (4.1.10) describes a pure state.)

Since (4.1.10) is the inevitable outcome of the process described and is *independent* of the initial state $\rho_0(o)$ of S_0, we have a rigorous quantum mechanical illustration of coercive preparation.

4.2. *Exact Evolution of the $S_0 + E_1$ Density Matrix*

The calculation of $\rho_{01}(t_1)$, the density matrix for $S_0 + E_1$ at the close of their interaction, is accomplished in two stages: (i) determination of the evolution operator and (ii) application of that operator to $\rho_0(o) \otimes \rho_1(o)$.

(i) *Evolution Operator for $S_0 + E_1$*

During the interval $[0, t_1]$ when S_0 interacts with E_1, the Hamiltonian for $S_0 + E_1$ is

$$H = H_0 + H_1 + V \qquad (4.2.1)$$

where V is defined by (4.1.5). Hence the evolution operator is

$$\exp(-it_1 H) = \exp[-it_1(H_0 + H_1 + V)] \qquad (4.2.2)$$

To find the matrix elements of (4.2.2) using the basis defined in (4.1.1), consider first the eigenvalue equation for H:

$$H|p\rangle_v = h_p|p\rangle_v, \quad p = 1\ldots, 4N \tag{4.2.3}$$

where the subscript v distinguishes the eigenkets of H from the common eigenkets $\{|m\rangle\}$ of H_0, H_1. Using (4.2.3) we obtain

$$\langle m|\exp(-it_1 H)|n\rangle = \sum_p \langle m|\exp(-it_1 H)|p\rangle_v {}_v\langle p|n\rangle$$

$$= \sum_p \langle m|p\rangle_v {}_v\langle p|n\rangle \exp(-it_1 h_p) \tag{4.2.4}$$

Thus to obtain the desired representation of the evolution operator, we must find the eigenvalues $\{h_p\}$ and the representations $\{\langle m|p\rangle_v\}$ of the eigenkets in the $\{|m\rangle\}$-representation. These are found by diagonalizing the H matrix. From (4.1.4) and (4.1.5), we get

$$\langle m|H|n\rangle = \langle m|H_0 + H_1 + V|n\rangle = \delta_{mn}\{2E[m \leqslant N] + (E+F)$$
$$\times [N < m \leqslant 3N] + 2F[3N < m]\}$$
$$+ v(1 - \delta_{mn})[N < m, n \leqslant 3N] \tag{4.2.5}$$

Because of the simple block structure of (4.2.5), we need only diagonalize $(1 - \delta_{mn})[N < m, n \leqslant 3N]$; consider the eigenvalue equation

$$\sum_{n=N+1}^{3N} (1 - \delta_{mn})\psi_n = g\psi_m \tag{4.2.6}$$

From (4.2.6) it follows that

$$\sum_{n=N+1}^{3N} \psi_n = (g+1)\psi_m \tag{4.2.7}$$

The left member of (4.2.7) is independent of m; hence $(a+1)\psi_m$ must be independent of m. Thus either $g = -1$ or $\psi_m = u$, a constant, where $u \neq 0$ since the null vector is inadmissible as an eigenvector.

If $\psi_m = u$, it follows from (4.2.7) that $g = 2N - 1$; i.e. $(2N - 1)$ is an eigenvalue of $(1 - \delta_{mn})[N < m, n \leqslant 3N]$ and the corresponding eigenvector has $2N$ equal components, ψ_m. The remaining $2N - 1$ eigenvectors belong to eigenvalue $g = -1$. From (4.2.7) it follows that the components ψ_m of each eigenvector belonging to $g = -1$ satisfy

$$\sum_{m=N+1}^{3N} \psi_m = 0 \tag{4.2.8}$$

One set of $2N - 1$ linearly independent vectors satisfying (4.2.8) has components

$$\{\psi_m^{(p)} = (\delta_{m,N+1} - \delta_{m,p+1})[N < m \leqslant 3N]$$
$$|p = N+1,\ldots, 3N - 1\} \tag{4.2.9}$$

From the set (4.2.9) an orthogonal set may be obtained by a straight-forward application of the Gram–Schmidt process; we finally obtain for·

the orthonormalized eigenvectors of the matrix $(1 - \delta_{mn})[N < m, n \leqslant 3N]$ the set

$$\left\{ \alpha_m^{(p)} = \frac{1 - \delta_{p,3N}}{\sqrt{[(p-N)(p-N+1)]}} \left[\sum_{q=N+1}^{p} \delta_{qm} - (p-N) \delta_{m,p+1} \right] \right.$$
$$\left. + \frac{1}{\sqrt{(2N)}} \delta_{p,3N} \quad p = N+1, \ldots, 3N \right\} \quad (4.2.10)$$

where $p = 3N$ belongs to eigenvalue $g = 2N - 1$ and $p = N + 1, \ldots, 3N - 1$ belong to eigenvalue $g = -1$. From (4.2.3), (4.2.5) and (4.2.10), it now follows that

$$\langle m|p \rangle_v = \delta_{mp}[m, p \leqslant N] + \alpha_m^{(p)}[N < m, p \leqslant 3N]$$
$$+ \delta_{mp}[3N < m, p] = {}_v\langle p|m \rangle \quad (4.2.11)$$

and

$$h_p = 2E[p \leqslant N] + (E + F - v)[N < p \leqslant 3N - 1]$$
$$+ [E + F + (2N - 1)v][p = 3N] + 2F[3N < p] \quad (4.2.12)$$

Substituting (4.2.11), (4.2.12) into (4.2.4) we obtain

$$\langle m|\exp(-it_1 H)|n \rangle = \sum_{p=1}^{4N} \langle m|p \rangle_v {}_v\langle p|n \rangle \exp(-it_1 h_p)$$

$$= \exp(-it_1 2E) \sum_{p=1}^{N} \delta_{mp} \delta_{np} + \exp[-it_1(E+F)] C_{mn}$$

$$+ \exp(-it_1 2F) \sum_{p=3N+1}^{4N} \delta_{mp} \delta_{np}$$

$$= \{\exp(-it_1 2E)[m, n \leqslant N] + \exp(-it_1 2F)$$
$$\times [3N < m, n]\} \delta_{mn}$$
$$+ \exp[-it_1(E+F)] C_{mn}[N < m, n \leqslant 3N] \quad (4.2.13)$$

where

$$C_{mn} = \exp(it_1 v) \sum_{p=N+1}^{3N-1} \alpha_m^{(p)} \alpha_n^{(p)} + \exp[-it_1(2N-1)v] \alpha_m^{(3N)} \alpha_n^{(3N)}$$

$$= \exp(it_1 v) \sum_{p=N+1}^{3N} \alpha_m^{(p)} \alpha_n^{(p)} + \{\exp[-it_1(2N-1)v] - \exp(it_1 v)\} \alpha_m^{(3N)} \alpha_n^{(3N)}$$

$$= \exp(it_1 v) \delta_{mn}[N < m, n \leqslant 3N] + \frac{1}{2N} \{\exp[-it_1(2N-1)v]$$
$$- \exp(it_1 v)\} \quad (4.2.14)$$

since from the completeness relation for the eigenvectors of the submatrix $(1 - \delta_{mn})[N < m, n \leqslant 3N]$ we have

$$\sum_{p=N+1}^{3N} \alpha_m^{(p)} \alpha_n^{(p)} = \sum_{p=N+1}^{3N} \langle m|p \rangle_v {}_v\langle p|n \rangle$$
$$= \langle m|n \rangle[N < m, n \leqslant 3N]$$
$$= \delta_{mn}[N < m, n \leqslant 3N] \quad (4.2.15)$$

Hence
$$C_{mn} = \exp(it_1 v)(X + \delta_{mn})[N < m, n \leqslant 3N] \qquad (4.2.16)$$
where
$$X \equiv \frac{1}{2N}[\exp(-it_1 2Nv) - 1]$$

The matrix elements of the evolution operator are therefore
$$\langle m|\exp(-it_1 H)|n\rangle = \langle n|\exp(it_1 H)|m\rangle^*$$
$$= \{e\theta(N + 1 - m) + f\theta(m - 3N)\}\,\delta_{mn}$$
$$+ y(X + \delta_{mn})[N < m, n \leqslant 3N] \quad (4.2.17)$$
where
$$\left\{ \begin{matrix} e \equiv \exp(it_1 2E), \quad f \equiv \exp(-it_1 2F) \\ y \equiv \exp[-it_1(E + F - v)] \end{matrix} \right\} \qquad (4.2.18)$$
and θ is the unit step symbol defined in (4.1.3).

(ii) *Dynamical Evolution from the Initial State* $\rho_0(o) \otimes \rho_1(o)$
Let the arbitrary matrix elements of $\rho_0(o)$ be symbolized as follows:
$$\left\{ \begin{matrix} {}_0\langle E|\rho_0(o)|E\rangle_0 \equiv a & {}_0\langle E|\rho_0(o)|F\rangle_0 \equiv c \\ {}_0\langle F|\rho_0(o)|E\rangle_0 = c^* & {}_0\langle F|\rho_0(o)|F\rangle_0 \equiv b \end{matrix} \right\} \qquad (4.2.19)$$
where a and b are real and nonnegative.

Using (4.1.1), (4.1.8), and (4.2.19) we then obtain the following form for the matrix elements of $\rho_0(o) \otimes \rho_1(o)$ in the $\{|m\rangle\}$ representation:
$$\langle p|\rho_0(o) \otimes \rho_1(o)|q\rangle$$
$$= \frac{1}{N}\{\delta_{pq}[a\theta(p - 2N) + (b - a)\,\theta(p - 3N)]$$
$$+ \delta_{p,q-N}\, c\theta(p - 2N) + \delta_{p,q+N}\, c^*\,\theta(p - 3N)\} \qquad (4.2.20)$$

Determination of $\rho_{01}(t_1)$ is now a straightforward but tedious computation.
$$\langle m|\rho_{01}(t_1)|n\rangle = \sum_{p=1}^{4N} \sum_{q=1}^{4N} \langle m|\exp(-it_1 H)|p\rangle\langle p|\rho_0(o)$$
$$\otimes \rho_1(o)|q\rangle\langle q|\exp(it_1 H)|n\rangle \qquad (4.2.21)$$

Expansion of (4.2.21) after substitution of (4.2.18) and (4.2.20) yields
$$\langle m|\rho_{01}(t_1)|n\rangle = \frac{1}{N}\sum_{pq} XX^*[N < m, n, p, q \leqslant 3N]$$
$$\times \delta_{pq}\{a\theta(p - 2N) + (b - a)\,\theta(p - 3N)\} \qquad (\alpha)$$
$$+ \frac{1}{N}\sum_{pq} XX^*[N < m, n, p, q \leqslant 3N]$$
$$\times \{c\delta_{p,q-N}\,\theta(p - 2N) + c^*\delta_{p,q+N}\,\theta(p - 3N)\} \qquad (\beta)$$

$$+ \frac{1}{N} \sum_{pq} X \delta_{qn} [N < m, n, p, q \leqslant 3N]$$
$$\times \delta_{pq} \{ a\theta(p - 2N) + (b - a)\,\theta(p - 3N) \} \tag{γ}$$

$$+ \frac{1}{N} \sum_{pq} X \delta_{qn} [N < m, n, p, q \leqslant 3N]$$
$$\times \{ c\delta_{p,q-N}\,\theta(p - 2N) + c^* \delta_{p,q+N}\,\theta(p - 3N) \} \tag{ϵ}$$

$$+ \frac{1}{N} \sum_{pq} X^* \delta_{mp} [N < m, n, p, q \leqslant 3N]$$
$$\times \delta_{pq} \{ a\theta(p - 2N) + (b - a)\,\theta(p - 3N) \} \tag{ζ}$$

$$+ \frac{1}{N} \sum_{pq} X^* \delta_{mp} [N < m, n, p, q \leqslant 3N]$$
$$\times \{ c\delta_{p,q-N}\,\theta(p - 2N) + c^* \delta_{p,q+N}\,\theta(p - 3N) \} \tag{η}$$

$$+ \frac{1}{N} \sum_{pq} \delta_{mp}\,\delta_{qn} [N < m, n, p, q \leqslant 3N]$$
$$\times \delta_{pq} \{ a\theta(p - 2N) + (b - a)\,\theta(p - 3N) \} \tag{ι}$$

$$+ \frac{1}{N} \sum_{pq} \delta_{mp}\,\delta_{nq} [N < m, n, p, q \leqslant 3N]$$
$$\times \{ c\delta_{p,q-N}\,\theta(p - 2N) + c^* \delta_{p,q+N}\,\theta(p - 3N) \} \tag{κ}$$

$$+ \frac{1}{N} \sum_{pq} y(X + \delta_{mp}) [N < m, p \leqslant 3N]$$
$$\times \delta_{pq} \{ a\theta(p - 2N) + (b - a)\,\theta(p - 3N) \}$$
$$\times \delta_{qn} \{ \theta(N + 1 - q)e^* + \theta(q - 3N)f^* \} \tag{ν}$$

$$+ \frac{1}{N} \sum_{pq} y^*(X^* + \delta_{qn}) [N < q, n \leqslant 3N]$$
$$\times \delta_{pq} \{ a\theta(p - 2N) + (b - a)\,\theta(p - 3N) \}$$
$$\times \delta_{mp} \{ \theta(N + 1 - m)e + \theta(m - 3N)f \} \tag{ξ}$$

$$+ \frac{1}{N} \sum_{pq} y(X + \delta_{mp}) [N < m, p \leqslant 3N]$$
$$\times \delta_{qn} \{ \theta(N + 1 - q)e^* + \theta(q - 3N)f^* \}$$
$$\times c^* \delta_{p,q+N}\,\theta(p - 3N) \tag{o}$$

$$+ \frac{1}{N} \sum_{pq} y(X + \delta_{mp})[N < m, p \leqslant 3N]$$

$$\times \delta_{qn}\{\theta(N + 1 - q) e^* + \theta(q - 3N) f^*\}$$

$$\times c \delta_{p,q-N}\, \theta(p - 2N) \tag{σ}$$

$$+ \frac{1}{N} \sum_{pq} \delta_{mp}\{\theta(N + 1 - m) e + \theta(m - 3N) f\}$$

$$\times y^*(X^* + \delta_{qn})[N < q, n \leqslant 3N]\, c \delta_{p,q-N}\, \theta(p - 2N) \tag{v}$$

$$+ \frac{1}{N} \sum_{pq} \delta_{mp}\{\theta(N + 1 - m) e + \theta(m - 3N) f\}$$

$$\times y^*(X^* + \delta_{qn})[N < q, n \leqslant 3N]\, c^* \delta_{p,q+N}\, \theta(p - 3N) \tag{ϕ}$$

$$+ \frac{1}{N} \sum_{pq} \delta_{mp} \delta_{qn}\{\theta(N + 1 - m) e + \theta(m - 3N) f\}$$

$$\times \{\theta(N + 1 - q) e^* + \theta(q - 3N) f^*\}$$

$$\times \delta_{pq}\{a\theta(p - 2N) - (a - b)\,\theta(p - 3N)\} \tag{χ}$$

$$+ \frac{1}{N} \sum_{pq} \delta_{mp} \delta_{qn}\, \theta(m - 3N)\, \theta(q - 3N)$$

$$\times \{c\delta_{p,q-N}\, \theta(p - 2N) + c^* \delta_{p,q+N}\, \theta(p - 3N)\} \tag{ω}$$

Because the bracketed index ranges and the conditions imposed by δ's and θ's are mutually exclusive, terms (β), (ϵ), (η), (κ), (ν), (ξ), (o), (v), and (ω) will vanish. For example, consider (v); after summing with the aid of the Kronecker δ's, we obtain

$$\frac{1}{N} y(X + \delta_{mn})[N < m, n \leqslant 3N]\{a\theta(n - 2N) + (b - a)\,\theta(n - 3N)\}$$

$$\times \{\theta(N + 1 - n) e^* + \theta(n - 3N) f^*\}$$

an expression that can be expanded into two terms one proportional to $\theta(N + 1 - n)[N < m, n \leqslant 3N]$, the other, to $\theta(n - 3N)[N < m, n \leqslant 3N]$. But n cannot satisfy both $n < N + 1$ and $n > N$, nor can n satisfy both $n > 3N$ and $n \leqslant 3N$. Hence the term (v) vanishes.

Similarly, parts of terms (α), (γ), (ζ), (ι), (σ), (ϕ), and (χ) vanish. When these simplifications are taken into account, the expansion (4.2.21) becomes

$$\langle m|\rho_{01}(t_1)|n\rangle = XX^* a[N < m, n \leqslant 3N] \tag{α}$$

$$+ \frac{1}{N} Xa[N < m \leqslant 3N][2N < n \leqslant 3N] \tag{γ}$$

$$+\frac{1}{N}X^* a[N < n \leqslant 3N][2N < m \leqslant 3N]$$
$$(\zeta)$$

$$+\frac{1}{N}\delta_{mn} a[2N < m \leqslant 3N]$$
$$(\iota)$$

$$+\frac{1}{N}yXf^*[N < m \leqslant 3N][3N < n \leqslant 4N]$$
$$(\sigma_1)$$

$$+\frac{1}{N}y^* X^* f^* [N < n \leqslant 3N][3N < m \leqslant 4N]$$
$$(\phi_1)$$

$$+\frac{1}{N}cy\,\delta_{m,n-N}f^*[3N < n \leqslant 4N]$$
$$(\sigma_2)$$

$$+\frac{1}{N}c^* y^* \delta_{n,m-N}f[3N < m \leqslant 4N]$$
$$(\phi_2)$$

$$+\frac{1}{N}b\,\delta_{mn} \tag{4.2.22}$$
$$(\chi)$$
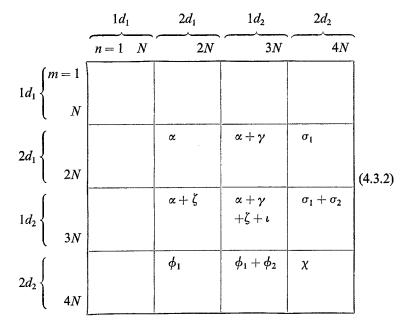
4.3. Sequence of Reduced Density Matrices for S_0

To determine $\rho_0(t_1)$, the trace of $\rho_{01}(t_1)$ over $\mathcal{H}_1$ must be calculated; the resultant matrix elements of $\rho_0(t_1)$ are given by

$$
\begin{aligned}
{}_0\langle E|\rho_0(t_1)|E\rangle_0 &= \sum_{d_1=1}^{N} \langle 1d_1|\rho_{01}(t_1)|1d_1\rangle \\
&\quad + \sum_{d_2=1}^{N} \langle 1d_2|\rho_{01}(t_1)|1d_2\rangle \\
{}_0\langle E|\rho_0(t_1)|F\rangle_0 = {}_0\langle F|\rho_0(t_1)|E\rangle_0^* &= \sum_{d_1} \langle 1d_1|\rho_{01}(t_1)|2d_1\rangle \\
&\quad + \sum_{d_2} \langle 1d_2|\rho_{01}(t_1)|2d_2\rangle \\
{}_0\langle F|\rho_0(t_1)|F\rangle_0 &= \sum_{d_1} \langle 2d_1|\rho_{01}(t_1)|2d_1\rangle \\
&\quad + \sum_{d_2} \langle 2d_2|\rho_{01}(t_1)|2d_2\rangle
\end{aligned}
\tag{4.3.1}
$$

where the symbols $\{|1d\rangle,\ |2d\rangle\}$ are related by (4.1.1), (4.1.2), to the set $\{|m\rangle\}$ used to represent $\rho_{01}(t_1)$ in (4.2.22).

To extract from (4.2.22) the quantities called for in (4.3.1), it is convenient to display the $\rho_{01}(t_1)$ matrix itself:

	$1d_1$	$2d_1$	$1d_2$	$2d_2$
	$n=1 \quad N$	$2N$	$3N$	$4N$
$1d_1$ ($m=1$ … N)				
$2d_1$ (… $2N$)		α	$\alpha+\gamma$	σ_1
$1d_2$ (… $3N$)		$\alpha+\zeta$	$\alpha+\gamma$ $+\zeta+\iota$	$\sigma_1+\sigma_2$
$2d_2$ (… $4N$)		ϕ_1	$\phi_1+\phi_2$	χ

$$(4.3.2)$$

The empty blocks in (4.3.2) contain only zeros; the Greek letters are keyed to the terms in (4.2.22). Studying (4.3.2) makes it easy to substitute (4.2.22) into (4.3.1) to obtain the reduced density matrix for S_0 at $t=t_1$.

$$(\rho_0(t_1)) \equiv \begin{pmatrix} {}_0\langle E|\rho_0(t_1)|E\rangle_0 & {}_0\langle E|\rho_0(t_1)|F\rangle_0 \\ {}_0\langle F|\rho_0(t_1)|E\rangle_0 & {}_0\langle F|\rho_0(t_1)|F\rangle_0 \end{pmatrix} = \begin{pmatrix} \mu_1 a & \lambda_1 c \\ \lambda_1{}^* c^* & (1-\mu_1)a+b \end{pmatrix}$$

$$(4.3.3)$$

where

$$\mu_1 \equiv NXX^* + X + X^* = 1 - \frac{1}{N}\sin^2 Nvt_1$$

and

$$\lambda_1 \equiv yf^*(1+X) = \exp\left[-i(E-F-v)t_1\right]\left\{1 + \frac{1}{2N}\left[\exp\left(-it_1 2Nv\right) - 1\right]\right\}$$

According to the discussion in Section 3, the reduced density matrix for S_0 after interaction with E_2 may be derived by considering the dynamical evolution of $S_0 + E_2$, with (4.3.3) taken as the state of S_0 at the beginning of the $S_0 - E_2$ interaction. The initial state of E_2 is the microcanonical ensemble density matrix (4.1.8).

Thus by exploiting the analogy between the $S_0 - E_1$ and $S_0 - E_2$ interactions, we conclude that the reduced density matrix for S_0 at $t = t_2$, the close of the $S_0 - E_2$ interaction, is given by

$$(\rho_0(t_2)) = \begin{pmatrix} \mu_2(\mu_1 a) & \lambda_2(\lambda_1 c) \\ \lambda_2^*(\lambda_1^* c^*) & (1 - \mu_2)(\mu_1 a) + (1 - \mu_1)a + b \end{pmatrix} \tag{4.3.4}$$

where

$$\mu_2 \equiv 1 - \frac{1}{N} \sin^2 Nv(t_2 - t_1)$$

and $\hspace{11cm}$ (4.3.5)

$$\lambda_2 \equiv \exp\left[-i(E - F - v)(t_2 - t_1)\right]\left[1 + \frac{1}{2N}\{\exp\left[-i(t_2 - t_1)2Nv\right] - 1\}\right]$$

From the normalization of $\rho_0(o)$, we have $a + b = 1$; hence the lower right element of (4.3.4) may be written as $1 - \mu_2\mu_1 a$.

Therefore, after S_0 has sequentially interacted with E_1, E_2, ..., E_K during the respective intervals $[0, t_1]$, $[t_1, t_2]$, ..., $[t_{K-1}, t_K]$, the reduced density matrix for S_0 will have evolved to the form

$$(\rho_0(t_K)) = \begin{pmatrix} \left(\prod_{k=1}^{K} \mu_k\right)a & \left(\prod_{k=1}^{K} \lambda_k\right)c \\ \left(\prod_{k=1}^{K} \lambda_k^*\right)c^* & 1 - \left(\prod_{k=1}^{K} \mu_k\right)a \end{pmatrix} \tag{4.3.6}$$

where

$$\mu_k \equiv 1 - \frac{1}{N}\sin^2 Nv\tau_k$$

and $\hspace{11cm}$ (4.3.7)

$$\lambda_k \equiv \exp\left[-i(E - F - v)\tau_k\right]\left\{1 + \frac{1}{2N}\left[\exp\left(-i\tau_k 2Nv\right) - 1\right]\right\}$$

with

$$\tau_k \equiv t_k - t_{k-1}$$

If each time interval τ_k has the same value τ, (4.3.6) becomes simply

$$(\rho_0(t_K)) = \begin{pmatrix} \mu^K a & \lambda^K c \\ \lambda^{*K} c^* & 1 - \mu^K a \end{pmatrix} \tag{4.3.8}$$

where μ and λ are given by (4.3.7) with $\tau_k = \tau$.

To prove that the interaction between S_0 and E drives S_0 to the final state (4.1.10), viz.,

$$(\rho_0(t_\infty)) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3.9}$$

we have only to prove that

$$\lim_{K\to\infty} \mu^K = 0 \qquad (4.3.10)$$

and

$$\lim_{K\to\infty} \lambda^K = 0 \qquad (4.3.11)$$

If $Nv\tau \neq l\pi$, $l = 0, \pm1, \pm2, \ldots$, then

$$|\mu| = \left|1 - \frac{1}{N}\sin^2 Nv\tau\right| < 1 \qquad (4.3.12)$$

From (4.3.12) it follows that (4.3.10) is valid for all choices of τ except a set of measure zero.

Similarly, to prove that (4.3.11) is correct, we note first that

$$|\lambda| = \left|1 + \frac{1}{2N}[\exp(-i\tau 2Nv) - 1]\right|$$

$$= \left\{\left[1 + \frac{\exp(-i\tau 2Nv) - 1}{2N}\right]\left[1 + \frac{\exp(i\tau 2Nv) - 1}{2N}\right]\right\}^{1/2}$$

$$= \left[1 - \frac{2}{N}\left(1 - \frac{1}{2N}\right)\sin^2 Nv\tau\right]^{1/2} \qquad (4.3.13)$$

Hence $|\lambda| < 1$, provided $Nv\tau \neq l\pi$, and (4.3.11) is therefore valid for almost all values of τ, the exceptions being the same set of measure zero for which (4.3.12) fails.

Since these results are independent of the $\rho_0(o)$ matrix elements a, b, c, we have demonstrated that the interactions described do indeed illustrate the concept of coercive preparation.

The rate of convergence of the state of S_0 towards the desired end depends on the magnitudes of $|\mu|$ and $|\lambda|$ and these in turn depend generally on the magnitude of τ in a periodic fashion. We can make this typically quantal situation appear more classical by choosing short enough times:

$$v\tau \ll 1/\sqrt{N} \qquad (4.3.14)$$

Then

$$|\mu| \to 1 - Nv^2\tau^2$$

and

$$|\lambda| \to 1 - \left(1 - \frac{1}{2N}\right)Nv^2\tau^2 \qquad (4.3.15)$$

The form (4.3.15) is obtained after the $S_0 - E_1$ interaction if second-order perturbation methods are used.

5. *The Preparing Environment: A Regression Problem*

The illustration of coercive preparation treated in detail above was intimately connected with the concept of energy; an interaction potential

energy was carefully adjusted so that the system would be driven toward an energy eigenstate, though not necessarily the ground state. Thus it is not obvious that similar coercive procedures can be devised for preparing states other than energy eigenstates of the system. However, the approach of Lamb, wherein the Hamiltonian is temporarily adjusted so that any pre-selected state becomes an energy eigenstate, offers one possible approach to the problem of generalizing our coercive preparation model to cover the preparation of arbitrary states.

More challenging, perhaps, than this matter of generalization is the question of preparing the environment E. In our model the initial state of the system to be prepared was of necessity left arbitrary but the preparing environment E had to have a particular initial density matrix dependent upon which state of S_0 was to be prepared by the interaction.

Hence we have a disconcerting infinite regression: how is the preparing environment E for S_0 to be prepared? If by interaction of E with an auxiliary preparing environment E', then how is E' prepared? Such queries can of course be pursued *ad infinitum*. Even Margenau's 'waiting' strategy is not immune to this objection: the electromagnetic field must be prepared and maintained in a pure vacuum state to ensure that a pure ground state eventually is produced for the system.

In the present paper we have simply bypassed this difficulty by appealing to equilibrium statistical mechanics, according to which a macro-system in thermodynamic equilibrium is properly described by the micro-canonical density matrix (4.1.8). In other words, we have assumed merely that macrosystems in thermodynamic equilibrium are readily available.

However, the very profound problem of relaxation to equilibrium (ergodic theory, H-theorems, etc.) is itself perhaps the most fundamental preparation-of-state problem in theoretical physics. Indeed it was the latter that originally motivated the present authors to undertake continuing investigations from which the present analysis of coercive preparation has emerged.

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